Calculate the limit

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$$\lim_{n\to\infty}\sum_{k=0}^{n}\frac{n^2}{2^n}\cdot\frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k}$$

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Since $\dfrac{k+4}{(k+1)(k+2)(k+3)} = \dfrac{1}{(k+1)(k+2)} + \dfrac{1}{(k+1)(k+2)(k+3)}$ then

$\dfrac{k+4}{(k+1)(k+2)(k+3)}\dbinom{n}{k} = \dfrac{1}{(k+1)(k+2)}\dbinom{n}{k} + \dfrac{1}{(k+1)(k+2)(k+3)}\dbinom{n}{k} =$

$\dfrac{1}{(n+1)(n+2)}\dbinom{n+2}{k+2} + \dfrac{1}{(n+1)(n+2)(n+3)}\dbinom{n+3}{k+3}$ and, therefore,

$\displaystyle\sum_{k=0}^{n}\frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k} = \frac{1}{(n+1)(n+2)}\sum_{k=0}^{n}\binom{n+2}{k+2} +$

$\displaystyle\frac{1}{(n+1)(n+2)(n+3)}\sum_{k=0}^{n}\binom{n+3}{k+3} = \frac{1}{(n+1)(n+2)}\sum_{k=2}^{n+2}\binom{n+2}{k} +$

$\displaystyle\frac{1}{(n+1)(n+2)(n+3)}\sum_{k=3}^{n+3}\binom{n+3}{k} = \frac{1}{(n+1)(n+2)}(2^{n+2}-n-3) +$

$\displaystyle\frac{1}{(n+1)(n+2)(n+3)}\left(2^{n+3}-n-4-\frac{(n+3)(n+2)}{2}\right) =$

$\dfrac{2^{n+2}-1}{(n+1)(n+2)} + \dfrac{2^{n+3}}{(n+1)(n+2)(n+3)} - \dfrac{1}{n+1} - \dfrac{1}{(n+1)(n+2)} - \dfrac{1}{2(n+1)}$

Noting that

$\displaystyle\lim_{n\to\infty}\left(\frac{n^2}{2^n}\cdot\frac{1}{(n+1)(n+2)}(2^{n+2}-n-3)\right) = \lim_{n\to\infty}\frac{4n^2}{(n+1)(n+2)} + \lim_{n\to\infty}\frac{n^2(n+3)}{2^n(n+1)(n+2)} =$

$4+0 = 4$ and $\displaystyle\lim_{n\to\infty}\left(\frac{n^2}{2^n}\cdot\frac{1}{(n+1)(n+2)(n+3)}\left(2^{n+3}-n-4-\frac{(n+3)(n+2)}{2}\right)\right) = 0$

($\displaystyle\lim_{n\to\infty}\frac{n^p}{2^n} = 0$ for any real p) we obtain $\displaystyle\lim_{n\to\infty}\sum_{k=0}^{n}\frac{n^2}{2^n}\cdot\frac{k+4}{(k+1)(k+2)(k+3)}\binom{n}{k} = 4.$